Homework on graphical models

1. Show that if $p(x, y, z) = g(x, z)h(y, z)$ for some nonnegative functions g and h, then $p(x, y|z) = p(x|z)p(y|z)$ (i.e., $X \perp Y \mid Z$).

2. In this exercise, you will derive the "separation criterion" for conditional independence in undirected graphical models. Suppose $p(x_1, \ldots, x_n)$ respects an undirected graph G. Suppose A, B, C are disjoint subsets of vertices of G, such that all paths from A to B are blocked by C.

   (a) Divide the set of vertices not in A, B, or C into two subsets: let D be the subset that can be reached from A without passing through C, and let E be all the rest. (Note that A, B, C, D, E are disjoint and, together, account for all the vertices.) Argue that for any clique Q, either $Q \cap D = \varnothing$ or $Q \cap E = \varnothing$, or both.

   (b) Show that $p(x_A, x_B|x_C) = p(x_A|x_C)p(x_B|x_C)$ (i.e., $X_A \perp X_B \mid X_C$). (Hint: Take the joint distribution, sum over x_D and x_E to get an expression for $p(x_A, x_B, x_C)$, apply (a) to split it into two factors, and then apply exercise 1 with X_A, X_B, X_C in place of X, Y, Z.)

3. Suppose p respects a DAG G. Let S be a subset of vertices, and let $\mathrm{an}(S)$ denote the set of ancestors of S (including S). Show that the marginal distribution $p(x_{\mathrm{an}(S)})$ respects the subgraph of ancestors of S (i.e., the graph obtained by removing any non-ancestors and their edges). (Hint: Take the joint distribution and sum out all non-ancestors.)

4. Show that if p respects a DAG G, then p also respects the (undirected) moralization of G.

5. In this exercise, you will derive the "moral ancestral separation criterion" for conditional independence in directed graphical models. Suppose p respects a DAG G. Suppose A, B, C are disjoint subsets of vertices of G, and let G_{MA} denote the moralization of the subgraph of ancestors of $S = A \cup B \cup C$. Argue that if all paths in G_{MA} from A to B are blocked by C, then $p(x_A, x_B|x_C) = p(x_A|x_C)p(x_B|x_C)$ (i.e., $X_A \perp X_B \mid X_C$). (Hint: Use exercises 3 and 4, along with the separation criterion for conditional independence in undirected graphical models.)