In-class exercise for Lecture 2

Instructions

- **Don't look at the solution yet!** This is for your benefit.
- This exercise must be submitted within 48 hours of the lecture in which it was given.
- As long as you do the exercise on time, you get full credit—your performance does not matter.
- Without looking at the solution, take 5 minutes to try to solve the exercise.
- Pre-assessment: Write down how correct you think your answer is, from 0 to 100%.
- Post-assessment: Now, study the solution and give yourself a "grade" from 0 to 100%.
- Submit your work on the course website, including the pre- and post- assessments.

Exercise

We write $X \sim \text{Poisson}(\theta)$ if X has the Poisson distribution with rate $\theta > 0$, that is, its p.m.f. is

$$p(x|\theta) = \text{Poisson}(x|\theta) = e^{-\theta}\theta^x/x!$$

for $x \in \{0, 1, 2, \ldots\}$ (and is 0 otherwise). Suppose $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Poisson}(\theta)$ given θ, and your prior is

$$p(\theta) = \text{Gamma}(\theta|a, b) = \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}\mathbb{1}(\theta > 0).$$

What is the posterior distribution on θ?

Solution

Since the data is independent given θ, the likelihood factors and we get

$$\begin{aligned} p(x_{1:n}|\theta) &= \prod_{i=1}^{n} p(x_i|\theta) \\ &= \prod_{i=1}^{n} e^{-\theta}\theta^{x_i}/x_i! \\ &\underset{\theta}{\propto} e^{-n\theta}\theta^{\sum x_i}. \end{aligned}$$

Thus, using Bayes' theorem,

$$\begin{aligned} p(\theta|x_{1:n}) &\propto p(x_{1:n}|\theta)p(\theta) \\ &\propto e^{-n\theta}\theta^{\sum x_i}\theta^{a-1}e^{-b\theta}\mathbb{1}(\theta > 0) \\ &\propto e^{-(b+n)\theta}\theta^{a+\sum x_i - 1}\mathbb{1}(\theta > 0) \\ &\propto \text{Gamma}\left(\theta \mid a + \textstyle\sum x_i,\, b+n\right). \end{aligned}$$

Therefore, since the posterior density must integrate to 1, we have

$$p(\theta|x_{1:n}) = \text{Gamma}\left(\theta \mid a + \textstyle\sum x_i,\, b+n\right).$$