In-class exercise

Instructions

- **Don't look at the solution yet!** This is for your benefit.
- This exercise must be submitted within 48 hours of the lecture in which it was given.
- As long as you do the exercise on time, you get full credit—your performance does not matter.
- Without looking at the solution, take 5 minutes to try to solve the exercise.
- Pre-assessment: Write down how correct you think your answer is, from 0 to 100%.
- Post-assessment: Now, study the solution and give yourself a "grade" from 0 to 100%.
- Submit your work on the course website, including the pre- and post- assessments.

Exercise

Suppose $X_1, \ldots, X_n$ are i.i.d. outcomes (heads or tails) from flipping a coin n times. You want to know whether the coin is fair (i.e., probability of heads is 1/2) or not. How would you approach this from a Bayesian hypothesis testing perspective? Give an explicit formula for the posterior on hypotheses.

Solution

This is a hypothesis testing problem with:

$$\begin{aligned}\mathrm{H}_0 &: \theta = 1/2\\ \mathrm{H}_1 &: \theta \neq 1/2.\end{aligned}$$

The Bayesian hypothesis testing approach is to put a prior on the two hypotheses, say, $p(\mathrm{H}_0) = \pi_0$ and $p(\mathrm{H}_1) = \pi_1 = 1 - \pi_0$, and

- given H_0, $\quad X_1, \dots, X_n \overset{\text{iid}}{\sim} \text{Bernoulli}(1/2)$,
- given H_1,

$$\begin{aligned}&\theta \sim \text{Beta}(a, b)\\ &X_1, \dots, X_n | \theta \overset{\text{iid}}{\sim} \text{Bernoulli}(\theta).\end{aligned}$$

The posterior on hypotheses is then

$$p(\mathrm{H}_0|x) = \frac{p(x|\mathrm{H}_0)p(\mathrm{H}_0)}{p(x|\mathrm{H}_0)p(\mathrm{H}_0) + p(x|\mathrm{H}_1)p(\mathrm{H}_1)}$$

and $p(\mathrm{H}_1|x) = 1 - p(\mathrm{H}_0|x)$, where

$$p(x|\mathrm{H}_0) = \prod_{i=1}^{n} \text{Bernoulli}(x_i|1/2) = (1/2)^n$$

and

$$\begin{aligned}p(x|\mathrm{H}_1) &= \int p(x|\theta, \mathrm{H}_1)p(\theta|\mathrm{H}_1)d\theta\\ &= \int \Big(\prod_{i=1}^{n} \text{Bernoulli}(x_i|\theta)\Big)\,\text{Beta}(\theta|a, b)d\theta\\ &= \frac{B(a + \sum x_i,\, b + n - \sum x_i)}{B(a, b)}.\end{aligned}$$

Hence,

$$p(\mathrm{H}_0|x) = \frac{(1/2)^n \pi_0}{(1/2)^n \pi_0 + \pi_1 B(a + \sum x_i,\, b + n - \sum x_i)/B(a, b)}.$$