In-class exercise

Instructions

- **Don't look at the solution yet!** This is for your benefit.
- This exercise must be submitted within 48 hours of the lecture in which it was given.
- As long as you do the exercise on time, you get full credit—your performance does not matter.
- Without looking at the solution, take 5 minutes to try to solve the exercise.
- Pre-assessment: Write down how correct you think your answer is, from 0 to 100%.
- Post-assessment: Now, study the solution and give yourself a "grade" from 0 to 100%.
- Submit your work on the course website, including the pre- and post- assessments.

Exercise

Suppose the data is modeled as i.i.d. $\text{Exp}(\theta)$, and the prior is

$$p(\theta) = \text{Gamma}(\theta|a, b) = \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}\mathbb{1}(\theta > 0).$$

We know that the posterior is

$$p(\theta|x_{1:n}) = \text{Gamma}(\theta|\alpha, \beta)$$

where $\alpha = a + n$ and $\beta = b + \sum_{i=1}^n x_i$.

What is the posterior predictive density $p(x_{n+1}|x_{1:n})$? Give your answer as a closed-form expression (not an integral).

Challenge

If you have any time left, can you also find the marginal likelihood $p(x_{1:n})$?

Solution

Denoting $x' = x_{n+1}$ for short, the posterior predictive is

$$
\begin{aligned}
p(x'|x_{1:n}) &= \int p(x'|\theta)p(\theta|x_{1:n})d\theta \\
&= \int_0^\infty \theta e^{-\theta x'} \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}d\theta \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \theta^{(\alpha+1)-1}e^{-(\beta+x')\theta}d\theta \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{(\beta+x')^{\alpha+1}} \int \mathrm{Gamma}(\theta \mid \alpha+1,\, \beta+x')d\theta \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{(\beta+x')^{\alpha+1}}.
\end{aligned}
$$

The marginal likelihood is

$$
\begin{aligned}
p(x_{1:n}) &= \int p(x_{1:n}|\theta)p(\theta)d\theta \\
&= \int_0^\infty \theta^n e^{-\theta\sum x_i} \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}d\theta \\
&= \frac{b^a}{\Gamma(a)} \int_0^\infty \theta^{a+n-1} \exp\big(-(b+\textstyle\sum x_i)\theta\big)d\theta \\
&= \frac{b^a}{\Gamma(a)} \frac{\Gamma(a+n)}{(b+\sum x_i)^{a+n}} \int \mathrm{Gamma}\big(\theta \mid a+n,\, b+\textstyle\sum x_i\big)d\theta \\
&= \frac{b^a}{\Gamma(a)} \frac{\Gamma(a+n)}{(b+\sum x_i)^{a+n}} = \frac{b^a}{\Gamma(a)} \frac{\Gamma(\alpha)}{\beta^\alpha}.
\end{aligned}
$$

The marginal likelihood can also be found by using Bayes' theorem: for any θ,

$$
p(x_{1:n}) = \frac{p(x_{1:n}|\theta)p(\theta)}{p(\theta|x_{1:n})} = \frac{\frac{b^a}{\Gamma(a)}\theta^{\alpha-1}e^{-\beta\theta}}{\mathrm{Gamma}(\theta|\alpha,\beta)} = \frac{\frac{b^a}{\Gamma(a)}}{\frac{\beta^\alpha}{\Gamma(\alpha)}}.
$$