

In-class exercise

Instructions

- **Don't look at the solution yet!** This is for your benefit.
- This exercise must be submitted within 48 hours of the lecture in which it was given.
- As long as you do the exercise on time, you get full credit—your performance does not matter.
- Without looking at the solution, take 5 minutes to try to solve the exercise.
- Pre-assessment: Write down how correct you think your answer is, from 0 to 100%.
- Post-assessment: Now, study the solution and give yourself a “grade” from 0 to 100%.
- Submit your work on the course website, including the pre- and post- assessments.

Exercise

Suppose $X_1, X_2, \dots \stackrel{\text{iid}}{\sim} p(x|\theta)$ given θ . Suppose the posterior predictive $p(x_{n+1}|x_{1:n})$ is too complicated to compute analytically, however, you can easily sample from the posterior $p(\theta|x_{1:n})$, and you can also easily compute the c.d.f.

$$F(c|\theta) = \mathbb{P}(X_i \leq c | \theta).$$

How would you construct a Monte Carlo approximation of

$$\mathbb{P}(X_{n+1} \leq c | x_{1:n})$$

for some given value of c ?

Solution

Approach 1 (Better)

$$\begin{aligned}
 \mathbb{P}(X^{n+1} \leq c \mid x_{1:n}) &= \int \mathbb{P}(X^{n+1} \leq c \mid \theta) d(\theta \mid x_{1:n}) \\
 &\approx \frac{1}{N} \sum_{i=1}^n \mathbb{P}(X^{n+1} \leq c \mid \theta_i) \\
 &= \frac{1}{N} \sum_{i=1}^n F(c \mid \theta_i)
 \end{aligned}$$

where $\theta_1, \dots, \theta_n \stackrel{iid}{\sim} p(\theta \mid x_{1:n})$. If you didn't understand the first line, note that by the law of iterated expectations, for any random variables X and Y ,

$$\mathbb{P}(X \leq c) = \mathbb{E}(\mathbb{1}(X \leq c)) = \mathbb{E}(\mathbb{E}(\mathbb{1}(X \leq c) \mid Y)) = \mathbb{E}(\mathbb{P}(X \leq c \mid Y)).$$

This can also be derived by writing out the integrals or sums for each of these expectations (and you should definitely do this if you didn't fully understand).

Approach 2

Another approach is to sample $(\theta_i, X_{(i)}^{n+1}) \sim p(\theta, x_{n+1} \mid x_{1:n})$ i.i.d. for $i = 1, \dots, N$, and use the approximation

$$\mathbb{P}(X^{n+1} \leq c \mid x_{1:n}) \approx \frac{1}{N} \sum_{i=1}^N \mathbb{1}(X_{(i)}^{n+1} \leq c).$$

These samples can be obtained by drawing $\theta_i \sim p(\theta \mid x_{1:n})$ and $X_{(i)}^{n+1} \sim p(x_{n+1} \mid \theta)$ independently for $i = 1, \dots, N$. However, this is probably not as good of an approximation as Approach 1, especially when $\mathbb{P}(X^{n+1} \leq c \mid x_{1:n})$ is very close to 0 or 1. The general principle is that if there is any part of the computation that can be done analytically, doing so will improve the approximation. (This is usually true, but not always.)