In-class exercise

Instructions

- **Don't look at the solution yet!** This is for your benefit.
- This exercise must be submitted within 48 hours of the lecture in which it was given.
- As long as you do the exercise on time, you get full credit—your performance does not matter.
- Without looking at the solution, take 5 minutes to try to solve the exercise.
- Pre-assessment: Write down how correct you think your answer is, from 0 to 100%.
- Post-assessment: Now, study the solution and give yourself a "grade" from 0 to 100%.
- Submit your work on the course website, including the pre- and post- assessments.

Exercise

You need to sample from the conditional distribution of $X \mid X < c$, where $X \sim \mathcal{N}(0,1)$ and $c \in \mathbb{R}$. Assume:

- you can generate Uniform$(0,1)$ random variables, and
- you can evaluate both the c.d.f. $F(x)$ and the inverse c.d.f. $F^{-1}(u)$ of the $\mathcal{N}(0,1)$ distribution.

How would you draw samples from $X \mid X < c$?

Solution

Approach 1 (Simple, but not great)

To draw a sample Z from the distribution of $X \mid X < c$,

1. sample $U \sim \text{Uniform}(0, 1)$,
2. set $X = F^{-1}(U)$,
3. if $X \geq c$ then return to step 1 (reject), otherwise, output $Z = X$ as a sample (accept).

Why does it work? By the inverse c.d.f. method, we know $X = F^{-1}(U) \sim \mathcal{N}(0, 1)$. By the rejection principle, if we reject any samples X such that $X \geq c$, then what remains has the conditional distribution given $X < c$. This approach is not ideal, however, since the rejection rate may be very high, especially when $c \ll 0$.

Approach 2 (Better)

To draw a sample Z from the distribution of $X \mid X < c$,

1. sample $U \sim \text{Uniform}(0, 1)$,
2. set $V = F(c)U$, and
3. set $Z = F^{-1}(V)$.

Why does this work? Note that in Approach 1, rejecting when $X \geq c$ is identical to rejecting when $U \geq F(c)$, and by the rejection principle, we know that the distribution of the U's that remain after rejection is $U \mid U < F(c)$, in other words, $\text{Uniform}(0, F(c))$. But that means that the rejection step can be bypassed completely by just sampling $V \sim \text{Uniform}(0, F(c))$ and setting $Z = F^{-1}(V)$! And we can directly sample $V \sim \text{Uniform}(0, F(c))$, by drawing $U \sim \text{Uniform}(0, 1)$ and setting $V = F(c)U$.