In-class exercise

Instructions

- **Don't look at the solution yet!** This is for your benefit.
- This exercise must be submitted within 48 hours of the lecture in which it was given.
- As long as you do the exercise on time, you get full credit—your performance does not matter.
- Without looking at the solution, take 5 minutes to try to solve the exercise.
- Pre-assessment: Write down how correct you think your answer is, from 0 to 100%.
- Post-assessment: Now, study the solution and give yourself a "grade" from 0 to 100%.
- Submit your work on the course website, including the pre- and post- assessments.

Exercise

You need to sample from the distribution with p.d.f.

$$p(x) \propto x^{a-1}\, \mathbb{1}(0 < x < b)$$

where $a, b > 0$. Assume you can generate $\text{Uniform}(0, 1)$ random variables. How would you draw samples from $p(x)$?

Solution

If we can get the c.d.f. and invert it, we can use the inverse c.d.f. method. First, let's find the normalizing constant of the p.d.f. For any $c > 0$,

$$\int_0^c x^{a-1} dx = \frac{x^a}{a}\Big|_0^c = \frac{c^a}{a}. \tag{0.1}$$

since $a > 0$. In particular, $\int_0^b x^{a-1} dx = b^a/a$, so

$$p(x) = \frac{a}{b^a} x^{a-1} \mathbb{1}(0 < x < b).$$

Thus, for $c \in (0, b)$, the c.d.f. is

$$\begin{aligned} F(c) &= \int_0^c p(x) dx \\ &= \int_0^c \frac{a}{b^a} x^{a-1} \mathbb{1}(0 < x < b) dx \\ &= \frac{a}{b^a} \int_0^c x^{a-1} dx \\ &= \frac{a}{b^a} \frac{c^a}{a} = (c/b)^a \end{aligned}$$

using Equation 0.1 again. To solve for F^{-1}, we set $u = F(x)$ for $u \in (0, 1)$ and solve for x:

$$\begin{aligned} u &= (x/b)^a \\ u^{1/a} &= x/b \\ bu^{1/a} &= x \end{aligned}$$

Thus, if $U \sim \text{Uniform}(0, 1)$ then $bU^{1/a} \sim p(x)$.