Homework 11

For reference on the Markov Chain material, see mathematicalmonk ML 18.1–18.9.

1. Make up an exercise on material from the second half of the course. Upload your exercise and the solution to the folder entitled "Student-created exercises (Final)" under Resources.

2. Consider a discrete distribution $p(u, v)$ on two variables u and v. Show that the Gibbs move of sampling u from $p(u|v)$ is a special case of a Metropolis–Hastings move with a particular proposal distribution.

3. For the following, consider only time-homogeneous Markov chains (MCs) on discrete spaces:

   (a) Give an example of a transition matrix that is not irreducible.

   (b) Give an example of a transition matrix that has more than one stationary distribution.

   (c) Give an example of a distribution π and a transition matrix T such that π is a stationary distribution for T, but π and T do not satisfy detailed balance.

   (d) Show that if a MC is irreducible and there is a state b such that $T_{bb} > 0$ (where T is the transition matrix), then the MC is aperiodic.

4. Let $\pi^{(0)} = (1, 0, 0)$, and compute $\pi^{(t)} = \pi^{(t-1)}T$ for $t = 1, \ldots, 8$, where

$$T = \begin{pmatrix} .7 & .2 & .1 \\ .1 & .7 & .2 \\ .2 & .1 & .7 \end{pmatrix}.$$

   (Report your results.) Find a distribution π such that $\pi T = \pi$. (Hint: It's easier to guess and verify than solve the equations.)

5. Consider the following model:

$$\begin{aligned} a &\sim \text{Gamma}(r, s) \\ b &\sim \text{Gamma}(u, v) \\ Y_1, \ldots, Y_n | a, b &\overset{\text{iid}}{\sim} \text{Gamma}(a, b). \end{aligned}$$

   Derive a MH-within-Gibbs sampler for the posterior $p(a, b|y_{1:n})$, using a Gibbs move for b and a Metropolis–Hastings move for a. For the MH move for a, generate proposals as follows: at iteration i, sample $X_i \sim \mathcal{N}(0, \sigma^2)$ and propose $a^* = a_i e^{X_i}$. To compute the acceptance ratio, you will need the density of this proposal distribution. To find it, use the formula for transformation of continuous random variables, namely, if $Y = g(X)$ then

$$p_Y(y) = p_X\big(g^{-1}(y)\big)\big|\tfrac{\partial}{\partial y} g^{-1}(y)\big|$$

(under regularity conditions on g) where p_X and p_Y are the densities of X and Y respectively. You will find that the proposal distribution is a log-normal with parameters that depend on a_i and σ.