Homework 3

Do exercises 2, 3, 5, and 7 at the end of Chapter 3 (Exponential Families and Conjugate Priors), and also the exercise below.

Exercise: Normal–Normal model

The normal (a.k.a. Gaussian) distribution $\mathcal{N}(\theta, \lambda^{-1})$ with mean θ and precision (inverse variance) $\lambda = 1/\sigma^2$ has p.d.f.

$$\mathcal{N}(x \mid \theta, \lambda^{-1}) = \sqrt{\frac{\lambda}{2\pi}} \exp\Big(-\frac{\lambda}{2}(x-\theta)^2\Big).$$

Suppose we have data $x_1, \ldots, x_n$ which we model as

$$X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathcal{N}(\theta, \lambda^{-1}).$$

Assume λ is fixed and known. Consider a normal prior on θ with mean μ_0 and precision λ_0, that is,

$$p(\theta) = \mathcal{N}(\theta \mid \mu_0, \lambda_0^{-1}) = \sqrt{\frac{\lambda_0}{2\pi}} \exp\Big(-\frac{\lambda_0}{2}(\theta-\mu_0)^2\Big).$$

Show that the posterior distribution is

$$p(\theta|x_{1:n}) = \mathcal{N}(\theta \mid M, L^{-1})$$

where $L = \lambda_0 + n\lambda$ and

$$M = \frac{\lambda_0\mu_0 + \lambda\sum_{i=1}^n x_i}{\lambda_0 + n\lambda}.$$

(Optional) For an extra challenge, can you find the marginal likelihood $p(x_{1:n})$?