Homework #1 (BST 263, Spring 2019)

Probability basics

Suppose X and Y are real-valued discrete random variables and $c \in \mathbb{R}$ is a (nonrandom) constant. Derive the following properties. Assume that variance is defined as $\mathrm{Var}(X) = \mathrm{E}((X - \mathrm{E}(X))^2)$. (Hint: Use the results of the previous problems to save yourself a lot of work. For example, you can use problems 1 and 2 to do problem 3.)

1. $\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y)$
2. $\mathrm{E}(cX) = c\mathrm{E}(X)$
3. $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2$
4. $\mathrm{Var}(X + c) = \mathrm{Var}(X)$
5. $\mathrm{Var}(cX) = c^2\mathrm{Var}(X)$
6. If X and Y are independent, then $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$.
7. $\mathrm{E}(\mathrm{E}(X|Y)) = \mathrm{E}(X)$

Linear algebra basics

Suppose $A, B \in \mathbb{R}^{n\times n}$. Derive the following properties.

8. $(AB)^{-1} = B^{-1}A^{-1}$, assuming A and B are invertible.

   (Hint: Just use the definition of the inverse of a matrix from the slides.)

9. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$
10. If A is SPSD then $x^\mathsf{T} Ax \geq 0$ for all $x \in \mathbb{R}^n$.

    (Use the definition of SPSD given in the slides.)

Random vectors

11. Show that if $Y, Z \in \mathbb{R}^n$ are independent random vectors, then $\mathrm{Cov}(Y + Z) = \mathrm{Cov}(Y) + \mathrm{Cov}(Z)$. (You may use the other properties of covariance matrices from the slides.)
12. Suppose you can generate independent standard normal random variables $Z_1, \ldots, Z_n$. Provide the formula for transforming these into a $\mathcal{N}(\mu, C)$ random vector, where $\mu \in \mathbb{R}^n$ and $C = B^\mathsf{T} B \in \mathbb{R}^{n\times n}$. Justify your answer using the properties from the slides.